Midterm Examination

Answer all questions. You should justify your answer and show all details.

1. (10 points) Let T be a triangle formed by the lines $x - y = 0, x + y = 3$ and the y-axis. Evaluate
$$\iint_T x \, dA(x, y) \ .$$

2. (10 points) Evaluate the integral
$$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) \, dx dy \ .$$

3. (15 points) Find the area enclosed by the cardioid $r = 1 + 2\cos\theta$.

4. (15 points) Let Ω be the region bounded by the planes $x + y + z = 1$ and $3x - 2y + z = 7$ in $x, y \geq 0$. Find the volume of Ω.

5. (15 points) Establish the following two formulas:

   (a)
$$\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \ .$$

   (b)
$$\int_0^\infty x^2 e^{-x^2} \, dx = \frac{\sqrt{\pi}}{4} \ .$$

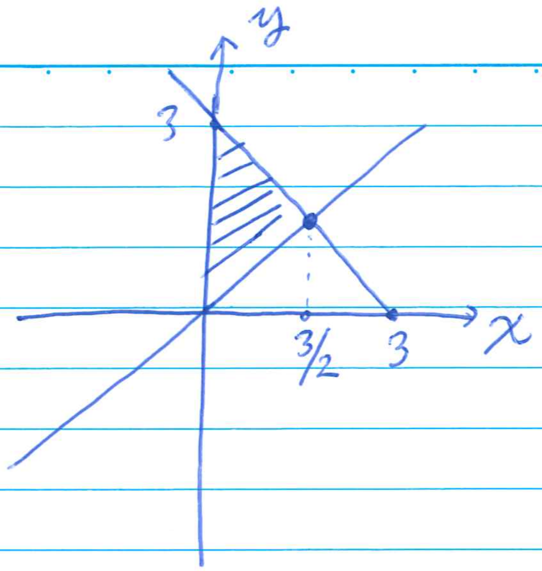
6. (10 points) Evaluate the iterated integral
$$\int_0^1 \int_y^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) \, dz dx dy$$
in a suitable coordinate system.

7. (15 points) Let Ω be the region lying between the planes $z = 0, 1$ and bounded by the parabola $z = 4 - x^2 - y^2$. For a function f defined in Ω, express $\iiint_\Omega f \, dV$ in (a) spherical coordinates and in (b) cylindrical coordinates.

8. (10 points) (a) Let D be a region of the form $\{(x, y) : \ -g(x) \leq y \leq g(x), \ a \leq x \leq b\}$ where g is a non-negative continuous function. Show that
$$\iint_D f(x, y) \, dA(x, y) = 0 \ ,$$
whenever f is a continuous function satisfying $f(x, -y) = -f(x, y)$ in D. (b) Suppose now D is a region symmetric with respect to the x-axis, that is, $(x, y) \in D$ implies $(x, -y) \in D$. Show that the conclusion in (a) still holds for f satisfying the same condition.

— Midterm Examination —
solution

#1 (10 pts)



$$\iint_T x\,dA = \int_0^{3/2}\int_x^{3-x} x\,dy\,dx$$

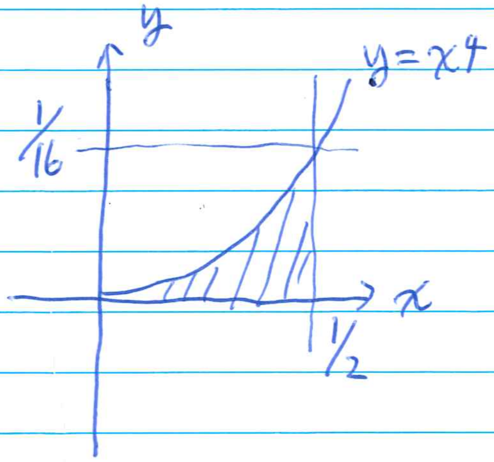
$$= \int_0^{3/2} xy\Big|_x^{3-x} dx$$

$$= \int_0^{3/2} x(3-x-x)\,dx = \left(\frac{3}{2}x^2 - \frac{2}{3}x^3\right)\Big|_0^{3/2}$$

$$= 9/8 \quad \#$$

(10 pts)

#2 Change the order of integration



$$I = \int_0^{\frac{1}{2}}\int_0^{x^4} \cos(16\pi x^5)\,dy\,dx$$

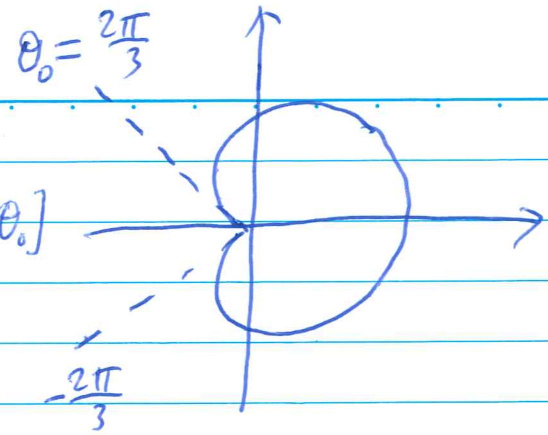
$$= \int_0^{\frac{1}{2}} \cos(16\pi x^5)\,x^4\,dx$$

$$= \frac{1}{5}\int_0^{\frac{1}{32}} \cos(16\pi t)\,dt$$

$$= \frac{1}{5}\,\frac{\sin(16\pi t)}{16\pi}\Big|_0^{\frac{1}{32}} = \frac{1}{5}\,\frac{\sin(\frac{\pi}{2})}{16\pi}$$

$$= \frac{1}{80\pi} \quad \#$$

3 (15 pts)



$$r = 1 + 2\cos\theta$$

need $r \geq 0$, it requires $\theta \in [-\theta_0, \theta_0]$

where

$$2\cos\theta_0 + 1 = 0$$

$$\cos\theta_0 = -\frac{1}{2}, \ \theta_0 \in (0, \pi)$$

$$\Rightarrow \theta_0 = 2\pi/3.$$

By symmetry, the area A:

$$\frac{1}{2} A = \int_0^{2\pi/3} \int_0^{1+2\cos\theta} r \, dr \, d\theta$$

$$\vdots$$

$$= 2\pi + \frac{3}{2}\sqrt{3} \quad \#$$

4 error

5 (a) see Lecture notes e.g. 1.30. (10 pts)

(b) $\displaystyle\int_0^a x^2 e^{-x^2} dx = \frac{-1}{2}\int_0^a x \, d(e^{-x^2})$ (5 pts)

$$= -\frac{1}{2} x e^{-x^2}\Big|_0^a + \frac{1}{2}\int_0^a e^{-x^2} dx$$

$$= -\frac{1}{2} a e^{-a^2} + \frac{1}{2}\int_0^a e^{-x^2} dx$$

Letting $a \to \infty$,

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-x^2} dx$$

$$= \sqrt{\pi}/4 \quad \text{by (a).}$$

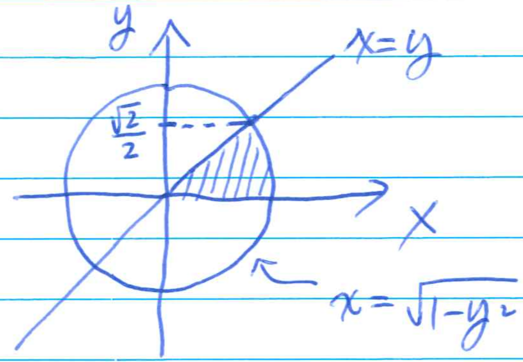
$\left(\lim_{a\to\infty} \frac{a}{e^{a^2}} \to 0 \ \text{L'Hospital}\right)$

\# 6 (10 pts) The region of integration is

$$\{(x,y,z) : 0 \le z \le x^2,\ (x,y) \in D\}$$ where

$$D = \{(x,y) : \quad y \le x \le \sqrt{1-y^2},\ y \in [0, \sqrt{2}/2]\}$$

D



$$\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} \int_0^{x^2} (x^2+y^2)\, dz\, dx\, dy$$

$$= \iint_D \int_0^{x^2} (x^2+y^2)\, dz\, dA(x,y)$$

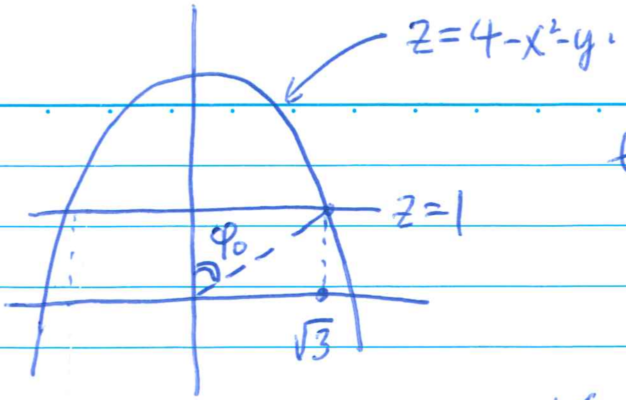
$$= \int_0^{\pi/4} \int_0^1 \int_0^{r^2\cos^2\theta} r^2\, dz\, r\, dr\, d\theta$$ (use cylindrincal coordinate)

$$= \int_0^{\pi/4} \int_0^1 r^5 \cos^2\theta\, dr\, d\theta$$

$$= \frac{1}{6} \int_0^{\pi/4} \cos^2\theta\, d\theta = \frac{1}{12} \int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{12} \left(\frac{\pi}{4} + \frac{1}{2} \sin 2\theta \Big|_0^{\pi/4} \right) = \frac{1}{12} \left(\frac{\pi}{4} + \frac{1}{2} \right). \ \#$$

#7



$\tan\varphi_0 = \sqrt{3} \Rightarrow \varphi_0 = \pi/3$

$z=1 \Leftrightarrow \rho\cos\varphi = 1 \Leftrightarrow \rho = \dfrac{1}{\cos\varphi}$.

ⓐ (10pts)

$$\iiint_\Omega f\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{1/\cos\varphi} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\cdot \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$+\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{\rho_0} f(\dots)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

Here $z=4-x^2-y^2$ turns to $\rho_0\cos\varphi = 4-\rho_0^2\sin^2\varphi$

So, $\rho_0(\varphi) = \dfrac{-\cos\varphi + \sqrt{1+15\sin^2\varphi}}{2\sin\varphi}$.

ⓑ (5pts)

$$\iiint_\Omega f\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{1} f(r\cos\theta, r\sin\theta, z)\,dz\, r\,dr\,d\theta$$

$$+\int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{4-r^2} f(r\cos\theta, r\sin\theta, z)\,dz\, r\,dr\,d\theta$$

#

Note: This problem is similar to Ex 1.16 in Notes.

#8 (a) (8 pts)

$$\iint_D f\,dA = \iint_{D+} f(x,y)\,dA + \iint_{D-} f(x,y)\,dA,$$

where $D_+ = D \cap \{z \geq 0\}$, $D_- = D \cap \{z \leq 0\}$

$$= \int_a^b \int_0^{g(x)} f(x,y)\,dy\,dx + \int_a^b \int_{-g(x)}^0 f(x,y)\,dy\,dx$$

$$= \int_a^b \int_0^{g(x)} f(x,y)\,dy\,dx + \int_a^b \int_{g(x)}^0 f(x,-t)(-dt)\,dx$$

$$= \int_a^b \int_0^{g(x)} f(x,y)\,dy\,dx + \int_a^b \int_{g(x)}^0 f(x,t)\,dt\,dx$$

$$= \int_a^b \int_0^{g(x)} f(x,y)\,dy\,dx - \int_a^b \int_0^{g(x)} f(x,t)\,dt\,dx$$

$$= 0.$$

(b) (2 pts) Extend f to $\tilde{f}$ by $\tilde{f}(x,y) = 0$ if $(x,y) \notin D$.
Then $\tilde{f}(x,-y) = -\tilde{f}(x,y)$.

Let R be a rectangle $[a,b] \times [-c,c]$ so that $D \subset R$.

Then $\iint_D f\,dA \overset{\text{def}}{=} \iint_R \tilde{f} = 0$ by (a) (Now $g(x) = c$)